NOTATION

r, the radius of the capillary; L, evaporation heat; A, atomic weight; T, temperature; R, gas constant; ξ, coordinate of the evaporation front; a, speed of sound; Π, porosity; λ, thermal conductivity; $\Phi^*(y) = (2/\sqrt{\pi}) \cdot \int_g^\infty \exp(-z^2)dz$; $v_n = v(0)$. The indices 1 and 2 pertain, respectively, to the parameters of the dry zone and of the initial body.

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APPLICATION OF THE TIKHONOV METHOD TO SOLVE THE INVERSE HEAT-CONDUCTION PROBLEM FOR A MELTING PLATE WITH MELT ENTRAINMENT

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An algorithm is proposed for the solution of the inverse heat-conduction problem for a melting plate with instantaneous removal of the liquid phase.

The inverse heat-conduction problem for a domain (plate) with moving boundary reduces to finding the law of motion of the melting solid body and two heat fluxes on the plate boundaries on the basis of the known temperature change in two interior points. The method of solving such a problem, proposed in one of the author's papers [1], is extended in this paper to the case when the method of successive intervals is inapplicable, i.e., when either the depth of the points x_1 and x_2 does not correspond to the magnitude of the time interval during which the temperatures are measured ($x_1^2/2l^2 > a\Delta\tau/l^2$ or $(l - x_2)^2/2l^2 > a\Delta\tau/l^2$) or the measurement errors are large $|\hat{t}(x_1, \tau) - t(x_1, \tau)|$. Underlying the method proposed is the more general approach to the solution of incorrect problems of mathematical physics proposed by Academician A. N. Tikhonov [2, 3].

Let us consider the temperature field in a plate heated by a heat flux of density $q_1(\tau)$, where the flux density $q_2(\tau)$ emerges through the opposite face of the plate.

Prior to the beginning of melting ($\tau < \tau_m$) this temperature field is subject to the heat-conduction equation

$$\lambda \frac{\partial^2 t}{\partial x^2} = c\rho \frac{\partial t}{\partial \tau}, \quad 0 \leqslant x \leqslant l, \quad 0 < \tau < \tau_m \tag{1}$$

with the boundary conditions

$$-\lambda \frac{\partial t}{\partial x}\bigg|_{x=0} = q_1(\tau), \tag{2}$$

$$-\lambda \frac{\partial t}{\partial x}\bigg|_{x=l} = -q_2(\tau) \tag{3}$$

and the initial condition

$$t(x, 0) = \varphi(x). \tag{4}$$

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The solution of (1) with the conditions (2)-(4) can be represented in the form

$$t(x, \tau) = c\rho \int_0^l G(x, \xi, \tau)\varphi(\xi)\,d\xi + \int_0^\tau G(x, 0, \tau-\eta)\,q_1(\eta)\,d\eta - \int_0^\tau G(x, l, \tau-\eta)\,q_2(\eta)\,d\eta, \quad (5)$$

where $G(x, \xi, \tau)$ is Green's function

$$G(x, \xi, \tau) = \frac{1}{c\rho}\cdot\frac{1}{2\sqrt{\pi a\tau}}\left\{\exp\left[-\frac{(x-\xi)^2}{4a\tau}\right] + \exp\left[-\frac{(x+\xi)^2}{4a\tau}\right] + \sum_{n=1}^{\infty}\left[\exp\left[-\frac{(2nl-x-\xi)^2}{4a\tau}\right] + \exp\left[-\frac{(2nl+x-\xi)^2}{4a\tau}\right] + \exp\left[-\frac{(2nl-x+\xi)^2}{4a\tau}\right] + \exp\left[-\frac{(2nl+x+\xi)^2}{4a\tau}\right]\right]\right\}, \quad (6)$$

$a = \lambda/c\rho$ is the thermal diffusivity of the plate.

If the behavior of the temperature with time is measured at two points of the plate $x = x_1$ and $x = x_2$ with the inevitable experimental error, then the heat fluxes $q_1(\tau)$ and $q_2(\tau)$ can be determined from equations of the type (5) by using the measured values $\hat{t}(x_1, \tau)$ and $\hat{t}(x_2, \tau)$.

An integral operator transform acting on the heat flux $q(\tau)$ in a symmetric integral operator is used in the Tikhonov method. Hence, we reduce the system of equations (5) for $x = x_1$ and $x = x_2$ to a form permitting execution of a similar transform. For this purpose, we perform a Laplace transform of the system (5) for the points $x = x_1$ and $x = x_2$, we extract the members $\overline{q_1}(p)$ and $\overline{q_2}(p)$ by multiplying both sides of the equation by $c\rho\lambda\sqrt{p/a}$, and we invert the Laplace transforms

$$\int_0^\tau \frac{q_1(\eta)}{\sqrt{\pi a(\tau-\eta)}}\sum_{n=0}^{\infty}\left[\exp\left(-\frac{[(2n-1)l-\Delta x]^2}{4a(\tau-\eta)}\right) - \exp\left(-\frac{[(2n+1)l+\Delta x]^2}{4a(\tau-\eta)}\right)\right]d\eta$$

$$= \frac{c\rho}{2}\int_0^\tau \frac{d\eta}{\sqrt{\pi a(\tau-\eta)^3}}\sum_{n=0}^{\infty}\left\{t(x_1, \eta)\left[[(2n+1)l-x_2]\exp\left(-\frac{[(2n+1)l-x_2]^2}{4a(\tau-\eta)}\right) + [(2n+1)l+x_2]\exp\left(-\frac{[(2n+1)l+x_2]^2}{4a(\tau-\eta)}\right)\right]\right.$$

$$\left. - t(x_2, \eta)\left[[(2n+1)l-x_1]\exp\left(-\frac{[(2n+1)l-x_1]^2}{4a(\tau-\eta)}\right) + [(2n+1)l+x_1]\exp\left(-\frac{[(2n+1)l+x_1]^2}{4a(\tau-\eta)}\right)\right]\right\}$$

$$- \frac{c\rho}{2}\int_0^{x_1}\frac{\varphi(\xi)}{\sqrt{\pi a\tau}}\sum_{n=0}^{\infty}\left[\exp\left(-\frac{[(2n+1)l-\Delta x-\xi]^2}{4a\tau}\right) + \exp\left(-\frac{[(2n+1)l-\Delta x+\xi]^2}{4a\tau}\right) - \exp\left(-\frac{[(2n+1)l+\Delta x-\xi]^2}{4a\tau}\right) - \exp\left(-\frac{[(2n+1)l+\Delta x+\xi]^2}{4a\tau}\right)\right]d\xi$$

$$- \frac{c\rho}{2}\int_{x_1}^{x_2}\frac{\varphi(\xi)}{\sqrt{\pi a\tau}}\sum_{n=0}^{\infty}\left[\exp\left(-\frac{[(2n+1)l-x_2-x_1+\xi]^2}{4a\tau}\right)\right.$$

$$+\exp\left(-\frac{[(2n+1)l-\Delta x+\xi]^2}{4a\tau}\right)-\exp\left(-\frac{[(2n+1)l+\Delta x-\xi]^2}{4a\tau}\right]$$

$$-\exp\left(-\frac{[(2n+1)l+x_2+x_1-\xi]^2}{4a\tau}\right)\Bigg]d\xi, \tag{7}$$

$$\int_0^\tau \frac{q_2(\eta)}{\sqrt{\pi a(\tau-\eta)}}\sum_{n=0}^{\infty}\left[\exp\left(-\frac{[(2n+1)l-\Delta x]^2}{4a(\tau-\eta)}\right)\right.$$

$$\left.-\exp\left(-\frac{[(2n+1)l+\Delta x]^2}{4a(\tau-\eta)}\right)\right]d\eta=$$

$$=\frac{c\rho}{2}\int_0^\tau \frac{1}{\sqrt{\pi a(\tau-\eta)^3}}\sum_{n=0}^{\infty}\left\{t(x_2,\eta)\left[(2nl+x_1)\exp\left(-\frac{(2nl+x_1)^2}{4a(\tau-\eta)}\right)\right.\right.$$

$$\left.+[(2n+2)l-x_1]\exp\left(-\frac{[(2n+2)l-x_1]^2}{4a(\tau-\eta)}\right)\right]$$

$$-t(x_1,\eta)\left[(2nl+x_2)\exp\left(-\frac{(2nl+x_2)^2}{4a(\tau-\eta)}\right)+[(2n+2)l\right.$$

$$\left.\left.-x_2]\exp\left(-\frac{[(2n+2)l-x_2]^2}{4a(\tau-\eta)}\right)\right]\right\}d\eta-\frac{c\rho}{2\sqrt{\pi a\tau}}\int_{x_2}^{l}\varphi(\xi)\sum_{n=0}^{\infty}\left[\exp\left(-\frac{(2nl-\Delta x+\xi)^2}{4a\tau}\right)\right.$$

$$+\exp\left(-\frac{[(2n+2)l-\Delta x-\xi]^2}{4a\tau}\right)-\exp\left(-\frac{(2nl+\Delta x+\xi)^2}{4a\tau}\right)$$

$$\left.-\exp\left(-\frac{[(2n+2)l+\Delta x-\xi]^2}{4a\tau}\right)\right]d\xi$$

$$-\frac{c\rho}{2\sqrt{\pi a\tau}}\int_{x_1}^{x_2}\varphi(\xi)\sum_{n=0}^{\infty}\left[\exp\left(-\frac{(2nl+x_2+x_1-\xi)^2}{4a\tau}\right)\right.$$

$$+\exp\left(-\frac{[(2n+2)l-\Delta x-\xi]^2}{4a\tau}\right)-\exp\left(-\frac{(2nl+\Delta x+\xi)^2}{4a\tau}\right)$$

$$\left.-\exp\left(-\frac{[(2n+2)l-\Delta x-\xi]^2}{4a\tau}\right)\right]d\xi, \tag{8}$$

where $\Delta x = x_2 - x_1$.

Substituting the temperatures $\hat{t}(x_1, \tau)$ and $\hat{t}(x_2, \tau)$, "perturbed" by errors into the system (7), (8) instead of $t(x_1, \tau)$ and $t(x_2, \tau)$ as well as the approximate distribution $\hat{\varphi}(x)$ in place of the exact value, we obtain a system of approximate integral equations whose solution can be executed by the Tikhonov method under the conditions

$$\frac{\partial q_1}{\partial \tau}(0)=\frac{\partial q_2}{\partial \tau}(0)=0.$$

After the beginning of the melting, the plate thickness which was initially equal to l, diminishes continuously because of entrainment of the melt by the incoming gas stream. Hence, the temperature field in the plate from the beginning of the melting $\tau = \tau_m$ will already not satisfy the expression (5). This field should satisfy the heat-conduction equation

$$\frac{\partial^2 t(x,\tau)}{\partial x^2}=\frac{1}{a}\cdot\frac{\partial t(x,\tau)}{\partial \tau},\quad s(\tau)\leqslant x\leqslant l, \tag{9}$$

with the boundary conditions

$$-\lambda\frac{\partial t}{\partial x}\bigg|_{x=s(\tau)}=q_1(\tau)-\rho L\frac{ds}{d\tau}, \tag{10}$$

$$-\lambda \frac{\partial t}{\partial x}\bigg|_{x=l} = -q_2(\tau), \tag{11}$$

$$t[s(\tau), \tau] = T_m, \tag{12}$$

$$s(\tau_m) = 0. \tag{13}$$

Here L is the specific heat of melting and $s(\tau)$ is the law of motion for the melting body surface.

As is known, the problem (9)-(13) is nonlinear and its analytic solution in general form is impossible. The approximate numerical solution of both the direct and inverse problems can be obtained as follows [1, 3-5].

We mentally continue the melting plate to its initial dimensions l. The solution of the heat-conduction equation (1), continued into the domain $[0, s(\tau)]$ will not satisfy conditions (2) and (3) since part of the energy brought to a plate of thickness l by the heat flux would be absorbed by the substance in the section $[0, s(\tau)]$ while the heat flux going to the boundary $s(\tau)$ would have a density less than $q_1(\tau)$.

Let us introduce an effective thermal flux density $Q_1(\tau)$:

$$Q_1(\tau) = q_1(\tau) + q_{1f}(\tau), \tag{14}$$

where $q_{1f}(\tau)$ is some fictitious flux, "the heat brought which is absorbed in the domain $[0, s(\tau)]$."

By using the effective flux $Q_1(\tau)$, the temperature field of the plate can be represented in the form (5), (7), and (8) after the beginning of melting with melt entrainment, where instead of $q_1(\tau)$ we should substitute $Q_1(\tau)$, where

$$Q_1(\tau) = \begin{cases} q_1(\tau) & \text{for} \quad \tau < \tau_m, \\ q_1(\tau) + q_{1f}(\tau) & \text{for} \quad \tau > \tau_m. \end{cases} \tag{15}$$

On the basis of the temperature values $\hat{t}(x_1, \tau)$ and $\hat{t}(x_2, \tau)$, where $x_1 > s(\tau)$, the integral equations (7) and (8) can be solved for the fluxes $Q_1(\tau)$ and $q_2(\tau)$ by the method of regularization.

Let us use (5) with condition (12) to determine the law of motion of the melting surface:

$$T_m = c\rho \int_0^l G[s(\tau), \xi, \tau]\varphi(\xi)d\xi + \int_0^\tau G[s(\tau), 0, \tau-\eta]Q_1(\eta)d\eta - \int_0^\tau G[s(\tau), l, \tau-\eta]q_2(\eta)d\eta. \tag{16}$$

The Green's functions in (16) are determined by iterating (6)

$$G[s_{N+1}^{(0)}, \xi, \tau] \cong G[s_N, \xi, \tau] + \frac{\partial G}{\partial x}\bigg|_{x=s_N}(s_{N+1}^{(0)} - s_N) + \frac{\partial^2 G}{\partial x^2}\bigg|_{x=s_N}\frac{(s_{N+1}^{(0)} - s_N)^2}{2}, \tag{17}$$

$$G(s_{N+1}^{(1)}, \xi, \tau) \cong G(s_{N+1}^{(0)}, \xi, \tau) + \frac{\partial G}{\partial x}\bigg|_{x=s_{N+1}^{(0)}}(s_{N+1}^{(1)} - s_{N+1}^{(0)}) + \frac{\partial^2 G}{\partial x^2}\bigg|_{x=s_{N+1}^{(0)}}\frac{(s_{N+1}^{(1)} - s_{N+1}^{(0)})^2}{2}, \tag{18}$$

etc., where $s_N = s(\tau_N)$, $s(0) = 0$, $\tau_N = N\Delta\tau$ are measured from the time of the beginning of melting τ_m, $\Delta\tau$ is the spacing of the difference mesh which is used to determine the heat fluxes $Q_1(\tau)$ and $q_2(\tau)$.

The true heat flux with density $q_1(\tau)$, which enters the plate through the melting surface, is determined from the Stefan condition (10) in which the rate of displacement of the melting surface is replaced by the difference analog

$$q_1(\tau_{N+1}) = \rho L \frac{s_{N+1} - s_N}{\Delta\tau} - \lambda c\rho \int_0^l \frac{\partial G}{\partial x}(s_{N+1}, \xi, \tau)\varphi(\xi)d\xi$$
$$-\lambda \int_0^\tau \frac{\partial G}{\partial x}(s_{N+1}, 0, \tau-\eta)Q_1(\eta)d\eta + \lambda \int_0^\tau \frac{\partial G}{\partial x}(s_{N+1}, l, \tau-\eta)q_2(\eta)d\eta. \tag{19}$$

This algorithm to solve the inverse problem is simpler than that proposed in [7] which uses the thermal potential. Extraction of the most awkward operations for determining the heat fluxes through the boundary of the initial domain (the true q_2 and effective Q_1) from iteratively finding the boundary location is achieved therein. Not used therein is the linear-fractional transformation of the variable size of the domain into a constant, which results in nonuniformity of the mesh.

However, the method proposed relies on an integral form of the solution of the heat-conduction equation (with Green's functions) and cannot be carried over to the nonlinear case. In the nonlinear case [$\lambda = \lambda(t)$, $c\rho = c(t)\rho(t)$] the algorithm proposed by Alifanov in later papers [8] should be used.

NOTATION

$t(x, \tau)$, true temperature at the point x of the plate at the time τ; $\hat{t}(x_i, \tau)$, temperature measured at the point x_i ($i = 1, 2$) at the time τ; λ, c, ρ, thermal conductivity, specific heat, and density of the plate material; l, plate thickness; x_1, x_2, interior points of the plate; $q_i(\tau)$, heat flux density through the plate surface; $G(x, \xi, \tau, \eta)$, Green's function; T_m, melting point; τ_m, time of the beginning of plate melting; L, specific heat of melting; $s(\tau)$, law of motion of the melted plate surface; q_{1f}, Q_{1f}, densities of the "fictitious" and "effective" fluxes (auxiliary quantities); $\Delta\tau$, difference mesh spacing; N, spacing number.

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DIRECT AND INVERSE HEAT-CONDUCTION PROBLEMS IN A MORE COMPLETE FORMULATION

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Direct and inverse heat-conduction problems are formulated and solved for the asymmetric cooling of an infinite plate with nonuniformly distributed and asynchronously acting sources in the case of an inhomogeneous initial distribution.

In electrical engineering, it is important to ensure that powerful electrical motors and generators will conform to the specified thermal operating conditions. In electronics, the construction and use of semiconductor devices ranging from powerful diodes to microcircuits also involves the optimization of temperature conditions of operation.

From a thermophysical point of view, this problem requires the development of experimental and theoretical methods of investigating the temperature in the cooling of a solid with internal sources. In electrical machines the appearance of heat sources is due to Joule-heat losses, remagnetization and eddy currents in magnetic and conducting parts of the machine, friction in the rotating parts, and losses in the circulation of the coolant gas. In semiconductors, heat liberation is due to Joule losses and the Peltier effect. Despite their

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